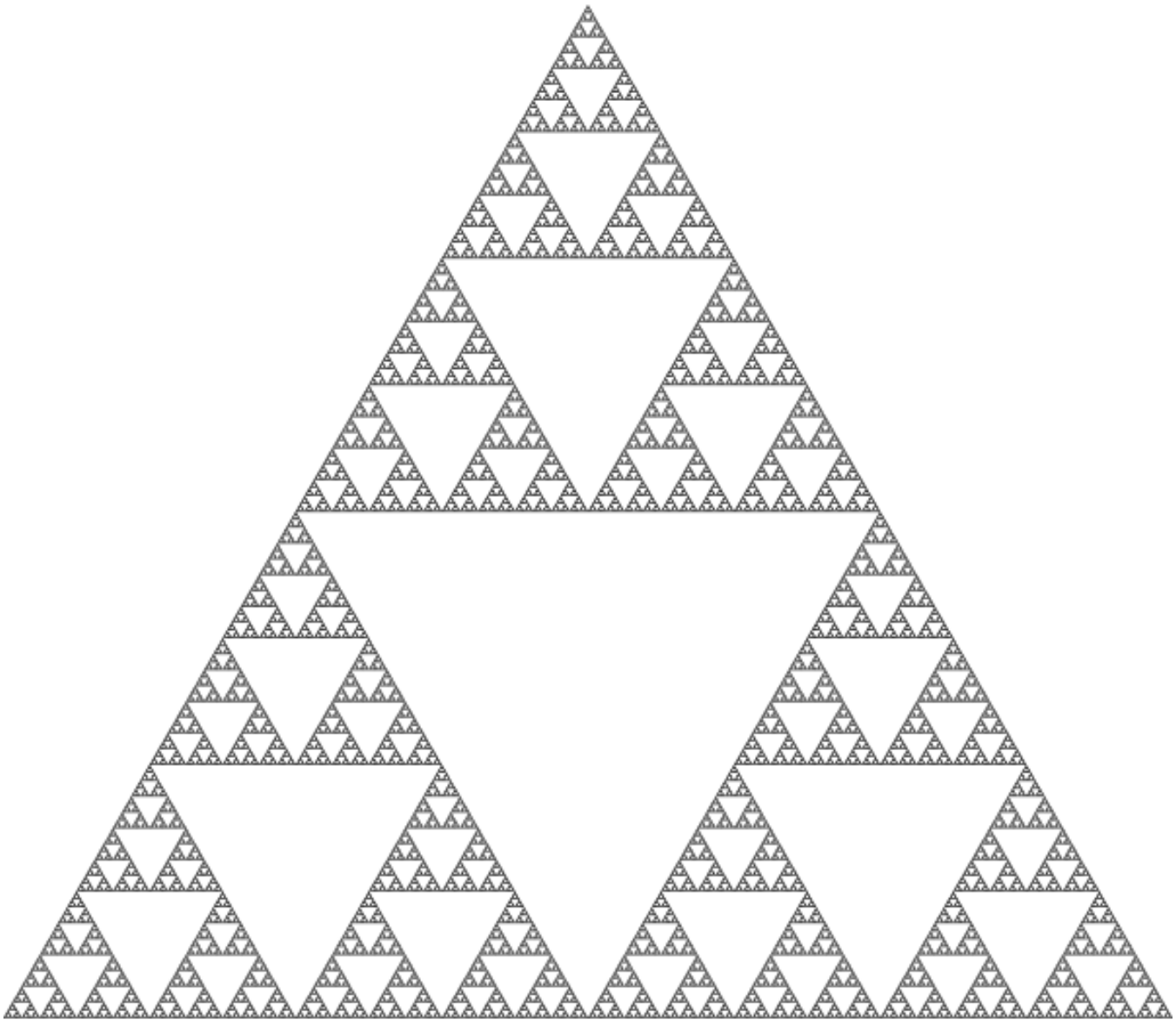
The Sierpinski Triangle



Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Table of Contents

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Introduction

In this CATE, the concept of self-similarity will be explored by a careful examination of the process required to develop the Sierpinski triangle. A figure is called self-similar when it can be decomposed into parts, where each part is a scaled version of the whole figure.

The first part of this CATE is devoted to the process applied to construct the Sierpinski triangle. Students will observe that, at each stage of construction, a figure is created consisting of three parts, each identical to the previous stage, but smaller. Students will discover that to create the next stage of the Sierpinski triangle, they have to reduce a copy of the previous stage, replicate it as many times as necessary, and reassemble the copies to form the new stage.

At a certain point in that process, it is impossible to distinguish between successive stages. The image obtained at this point is called the limit image of the Sierpinski triangle. Since all preliminary stages consist of smaller versions of the previous stage, self-similarity appears only in the limit image.

As the process of reduction, replication and reassembly proceeds, students will investigate how the perimeter and the area of the figures change through the various stages. Students will discover that the perimeter is increasing from step to step while the total area is decreasing at the same time. The limit image of the Sierpinski triangle is a figure with an infinite perimeter and zero area.

For the subtriangles in the different stages of the Sierpinski triangle, an addressing system will be introduced. Students will discover that the subtriangles can be labeled hierarchically according to the process leading from stage to stage in the construction of the Sierpinski triangle.

Finally, this CATE will move into non-planar development of the Sierpinski tetrahedron, which is again a self-similar object arising from a step-by-step process. In each step of the construction, four copies of a reduced version of the previous stage can be fit together in order to arrive at the new stage. Students will construct and examine the first three stages of the tetrahedron and examine the "hole" created inside the tetrahedron at Stage 1. The student will investigate the total surface area of the figure, the total volume of the figure, and the total length of the edges of the figure.

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Unit 1 Finding Patterns in an Iterative Process

Building the Sierpinski Triangle

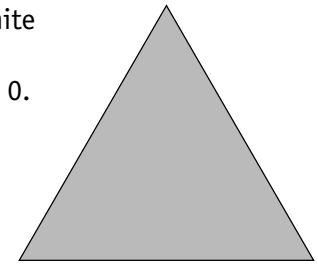
In Unit 1, students will explore the iterative process used to form successive stages of the Sierpinski triangle. This exploration will serve as an introduction to the idea of self-similarity which will be explored further in Unit 2. Students will discover how the perimeter and area of the figures generated by the iterative process change at each stage. This activity should be completed with the students working in pairs.

Grade Level: 5 - 8

Materials: Packet of template sheets containing:
- One 8″ shaded triangle
- Three 8″ white triangles
- 4″ shaded triangles
- 2″ shaded triangles
- 1″ shaded triangles

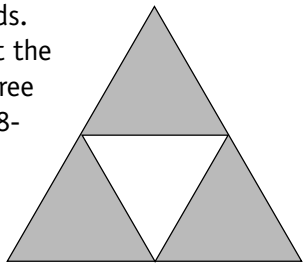
Pencil
Scissors
Glue stick
Perimeter of the Sierpinski Triangle transparency
Finding Patterns in the Sierpinski Triangle handout
Finding Patterns in the Sierpinski Triangle transparency

1. Distribute the packet of template sheets. Have the students cut out the one 8-inch shaded and the three 8-inch white equilateral triangles. Keep the shaded triangle on the desktop for reference. This will be referred to as Stage 0.



Stage 0

2. Display the Stage 1 triangle for approximately 10 seconds. Using the materials they have, ask students to construct the figure they have just seen. [Students should cut out three 4-inch shaded triangles and glue them onto one of the 8-inch white triangles.] Keep the Stage 1 triangle on the desktop for reference.



Stage 1

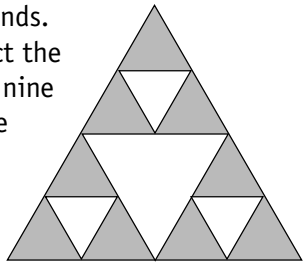
SSS

Sunshine State Standards

MA.A.2.3.1
MA.A.5.3.1
MA.B.12.1
MA.B.1.3.1
MA.B.1.3.3
MA.B.2.2.2
MA.C.1.3.1
MA.C.2.2.1
MA.C.2.3.1
MA.D.1.2.1
MA.D.1.2.2
MA.D.1.3.1
LA.A.2.2.8
SC.H.1.2.2
SC.H.1.2.5
SC.H.1.3.5
SC.H.2.3.1

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

3. Display the Stage 2 triangle for approximately 20 seconds. With the materials they have, ask students to construct the figure they have just seen. [Students should cut out nine of the 2-inch shaded triangles and glue them onto one of the 8-inch white triangles.]

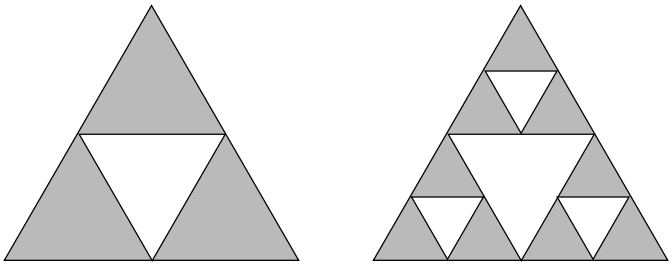


Stage 2

Exploring Similarity

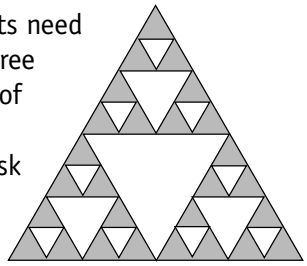
4. Ask the students to examine the three figures: Stage 0, Stage 1, and Stage 2. Begin a discussion by asking the students, "How are they the same?" "How are they different?" [Same: The size of each figure is the same. Within each figure the size of the triangles are the same. Different: The number of shaded triangles in each figure increases; however, the size of the triangles in each figure decreases. Each of the parts is identical to the previous stage but smaller.]

5. Display the *Perimeter in the Sierpinski Triangle* transparency. Discuss that each stage of the construction yields a figure with an area and a perimeter. Let the students discover that the perimeter in a certain stage can be composed from all sides of all shaded triangles.



6. Distribute the *Finding Patterns in the Sierpinski Triangle* handout. Explain to the students that the first 8-inch shaded triangle is the initial stage of the Sierpinski triangle referred to as Stage 0. Stage 0 has a side of length "s" and an area of "A". The next two figures the students constructed are referred to as Stage 1 and Stage 2, respectively. Together with the students, complete the table for Stages 0, 1, and 2.

7. With the materials provided, ask the students to construct their prediction of the Stage 3 figure without gluing. Allow students time to explore and struggle with this task. If students need assistance, remind them that Stage 2 is made up of three smaller Stage 1 figures; therefore, Stage 3 is made up of three smaller Stage 2 figures. When students have arranged the shaded triangles on the white triangle, ask them to explain how they arrived at their configuration. Before gluing, ensure that students have placed twenty-seven of the 1-inch triangles in the correct position on the white triangle. Continue the discussion of "What is the same?" and "What is different?" and complete the table for Stage 3.



Stage 3

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

8. At this point, it is important to summarize and emphasize the following points:
   - Each figure was constructed by a step-by-step iterative process.
   - Each figure is made up of three identical parts.
   - Each part looks like the previous stage but smaller.

Finding Patterns in the Sierpinski Triangle

9. Ask students to complete the table for Stage 4. Upon completion of the row for Stage 4, discuss the students' answers to verify that students have filled in the row correctly and determined the pattern. Together with the students, complete the table for Stage n by examining the patterns of change that occur in each column.

10. Discuss the amazing fact that the perimeter of this figure is increasing while the area is decreasing! This fact contradicts the relationship of perimeter and area that the students have encountered in the past.

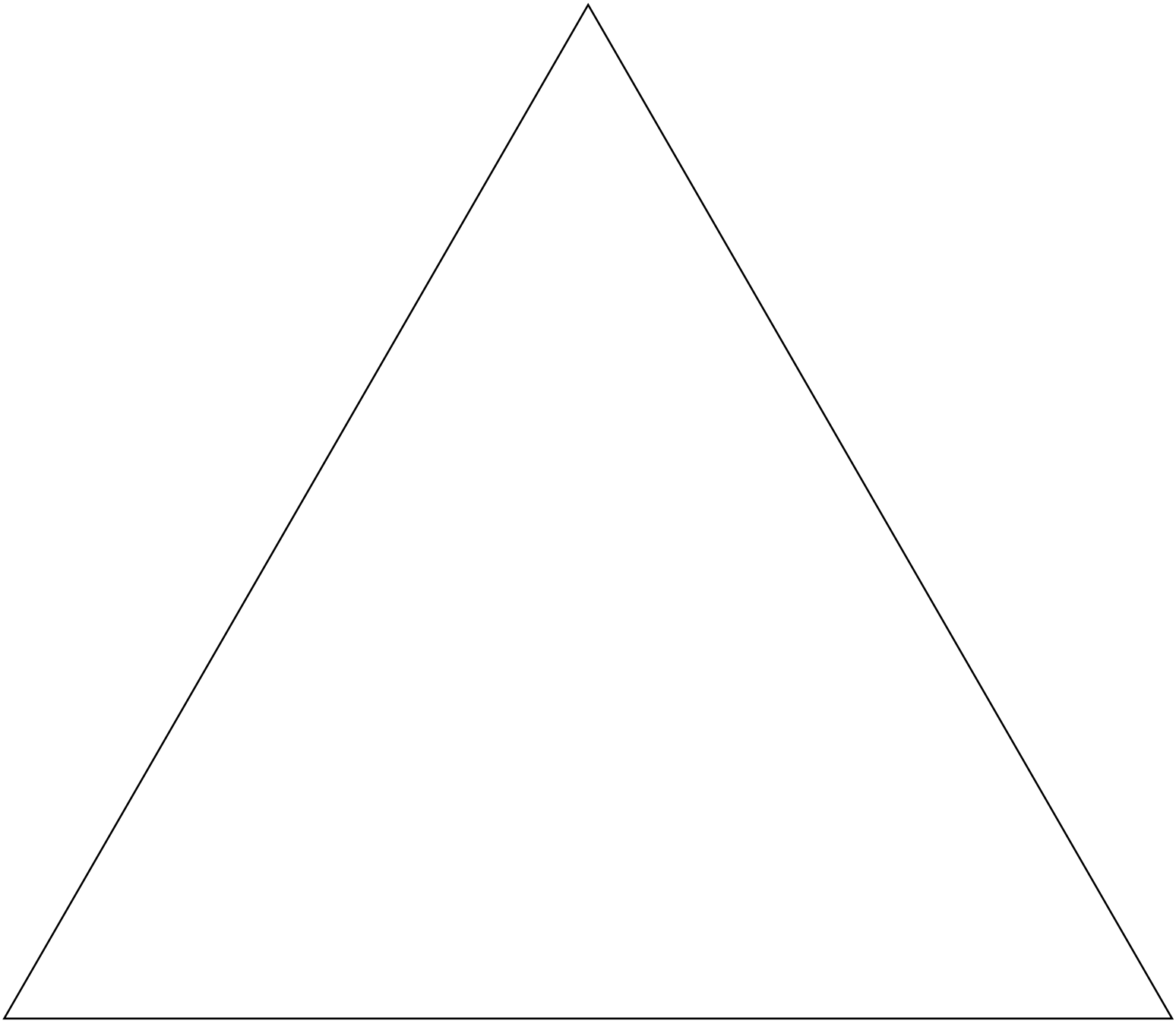
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Finding Patterns in the Sierpinski Triangle

Stage	Number of shaded triangles	Length of side of one shaded triangle	Total length of all sides of all shaded triangles (perimeter)	Area of one shaded triangle	Total area of all shaded triangles
0					
1					
2					
3					
4					
⋮	⋮	⋮	⋮	⋮	⋮
n					

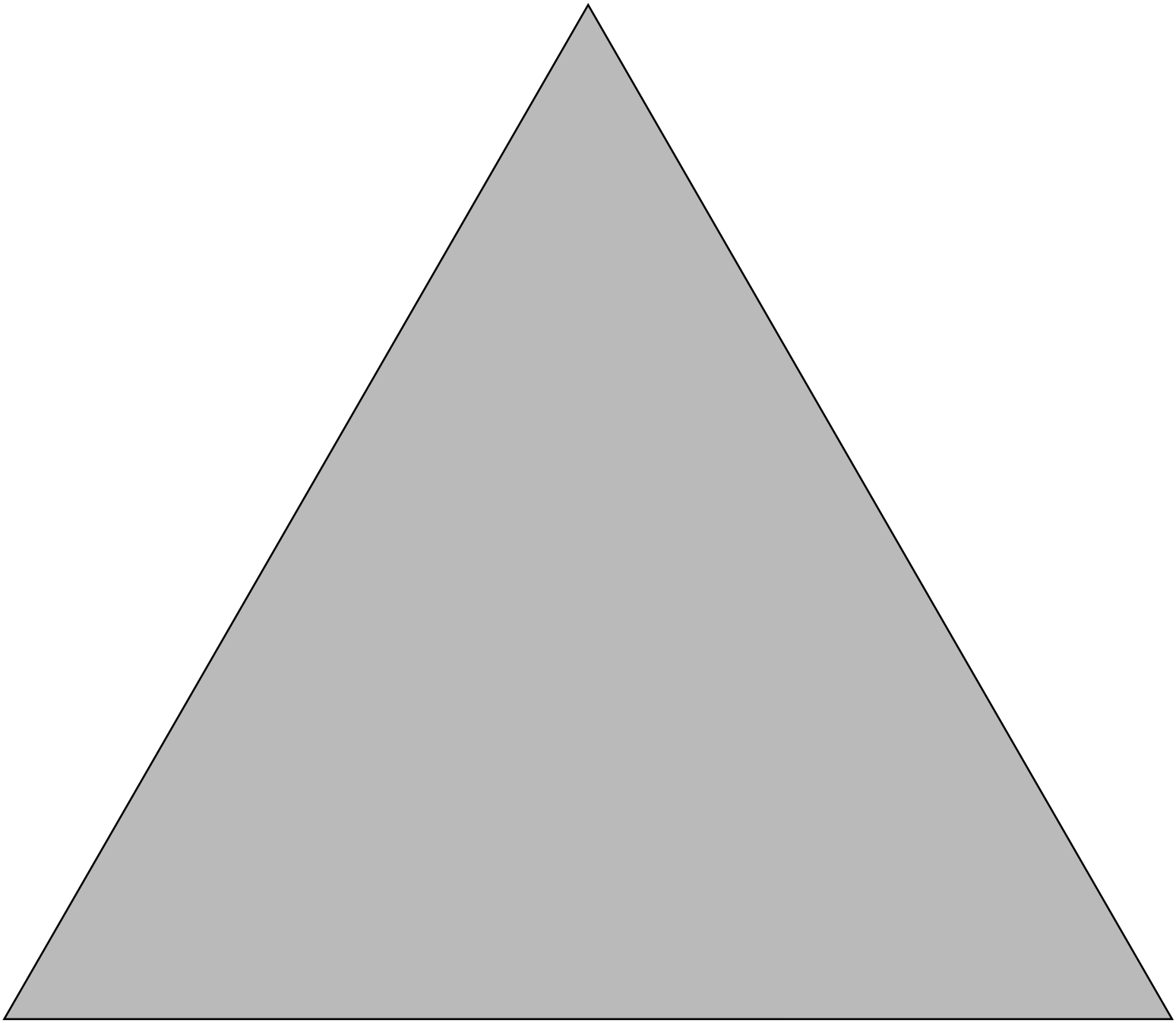
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

8-inch White Triangle Template



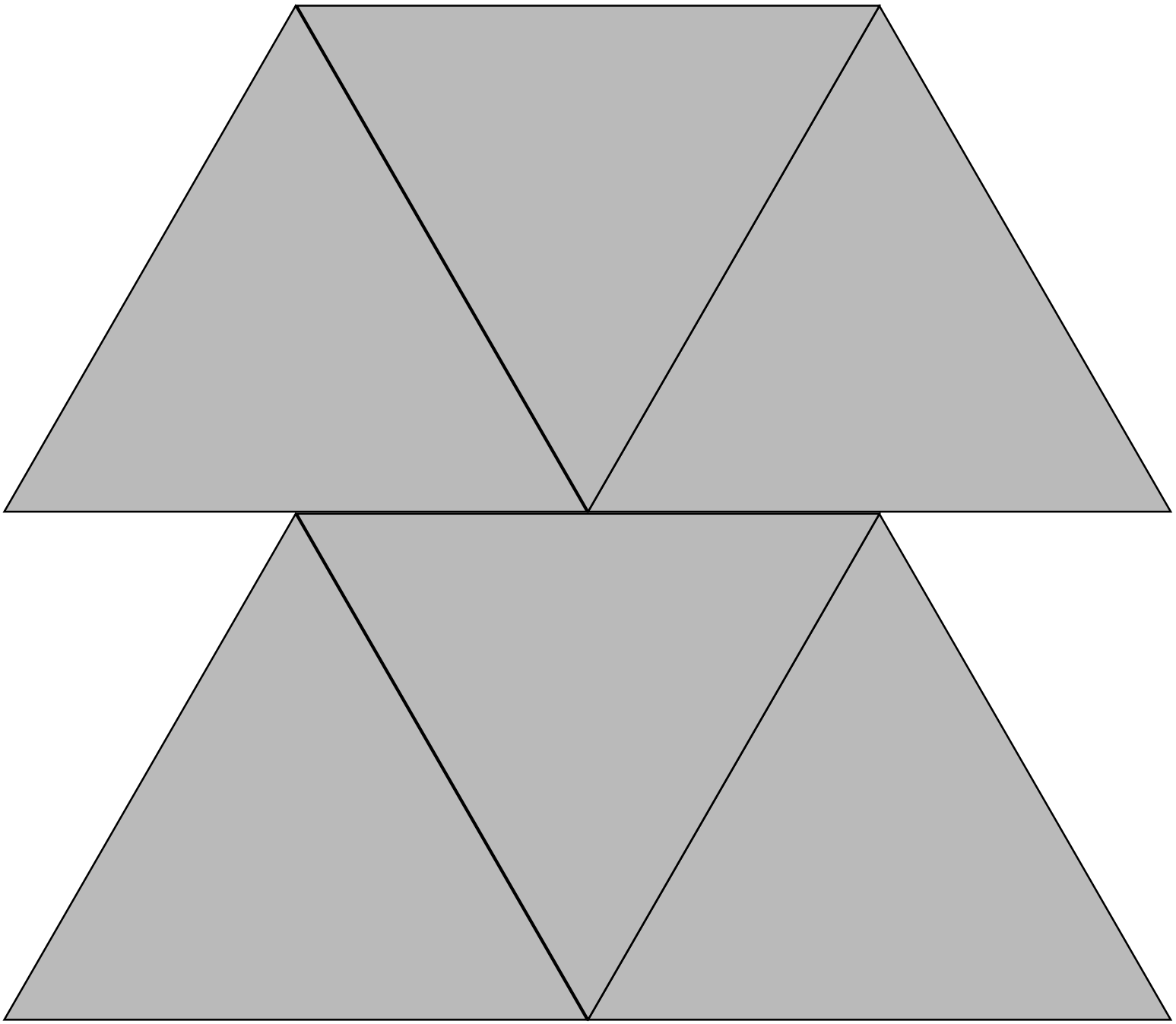
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

8-inch Shaded Triangle Template



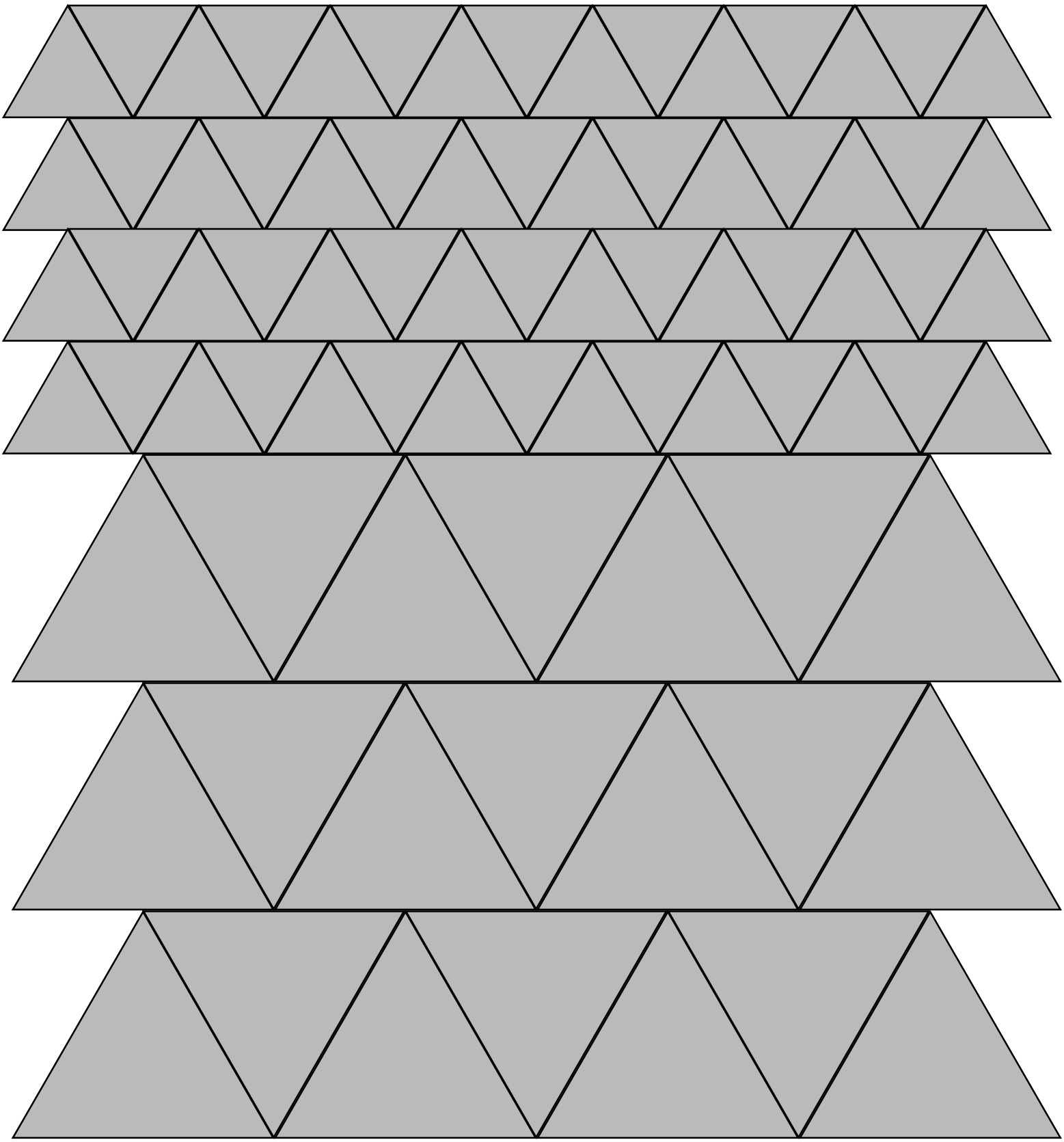
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

4-inch Triangle Template



Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

2-inch and 1-inch Triangle Template



Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Answer Keys for
The Sierpinski Triangle Unit 1

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Finding Patterns in the Sierpinski Triangle

Stage	Number of shaded triangles	Length of side of one shaded triangle	Total length of all sides of all shaded triangles (perimeter)	Area of one shaded triangle	Total area of all shaded triangles
0	1	s	$3 \cdot s$	A	A
1	3	$\left(\frac{1}{2}\right)s$	$9 \cdot \left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)A$	$3 \cdot \left(\frac{1}{4}\right)A$
2	9	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$27 \cdot \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$	$9 \cdot \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$
3	27	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$81 \cdot \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$	$27 \cdot \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$
4	81	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$243 \cdot \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$	$81 \cdot \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	3^n	$\left(\frac{1}{2}\right)^n s$	$3 \cdot 3^n \cdot \left(\frac{1}{2}\right)^n s$	$\left(\frac{1}{4}\right)^n A$	$3^n \cdot \left(\frac{1}{4}\right)^n A$

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Blackline Masters for
The Sierpinski Triangle Unit 1

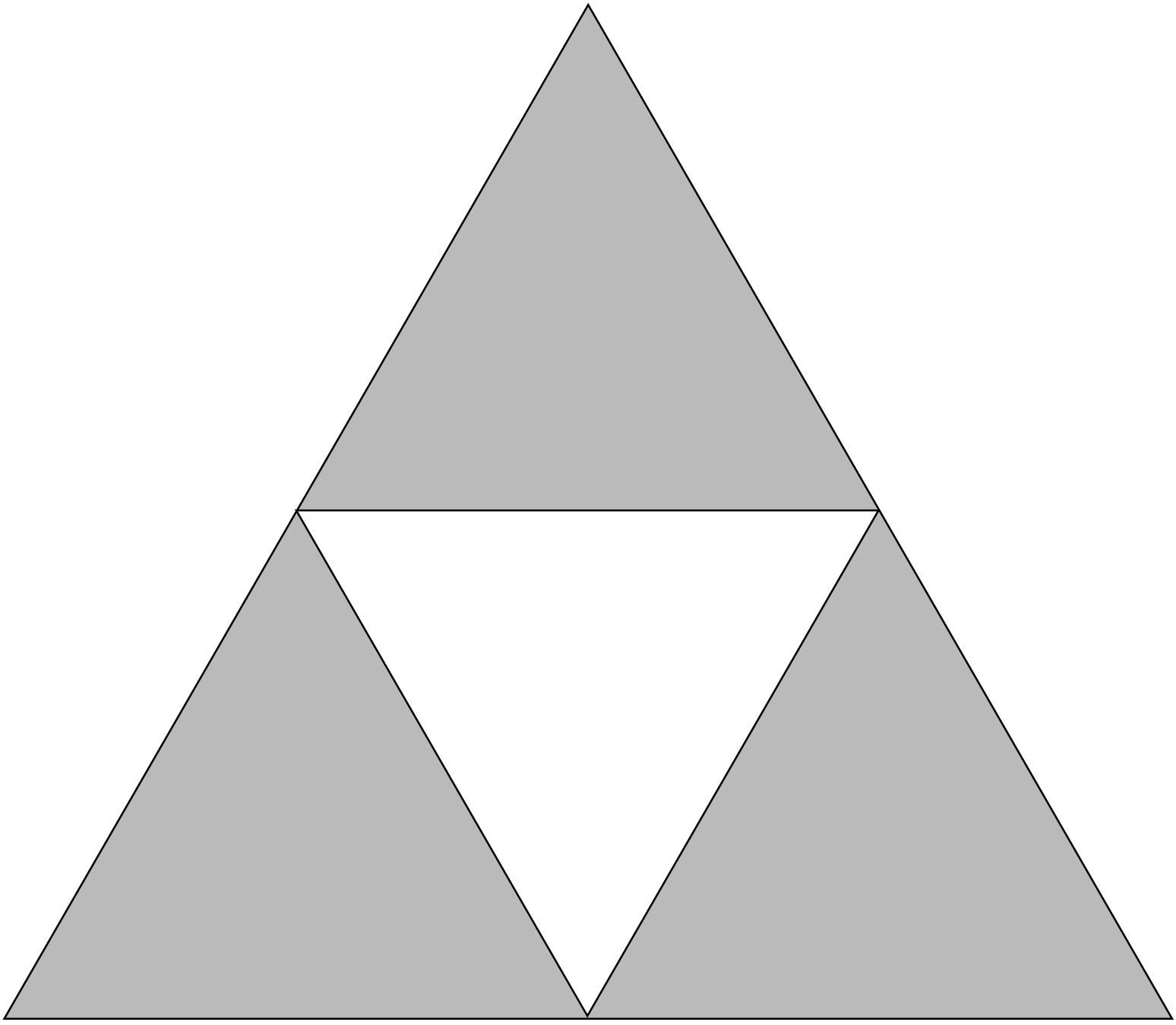
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Finding Patterns in the Sierpinski Triangle

Stage	Number of shaded triangles	Length of side of one shaded triangle	Total length of all sides of all shaded triangles (perimeter)	Area of one shaded triangle	Total area of all shaded triangles
0					
1					
2					
3					
4					
⋮	⋮	⋮	⋮	⋮	⋮
n					

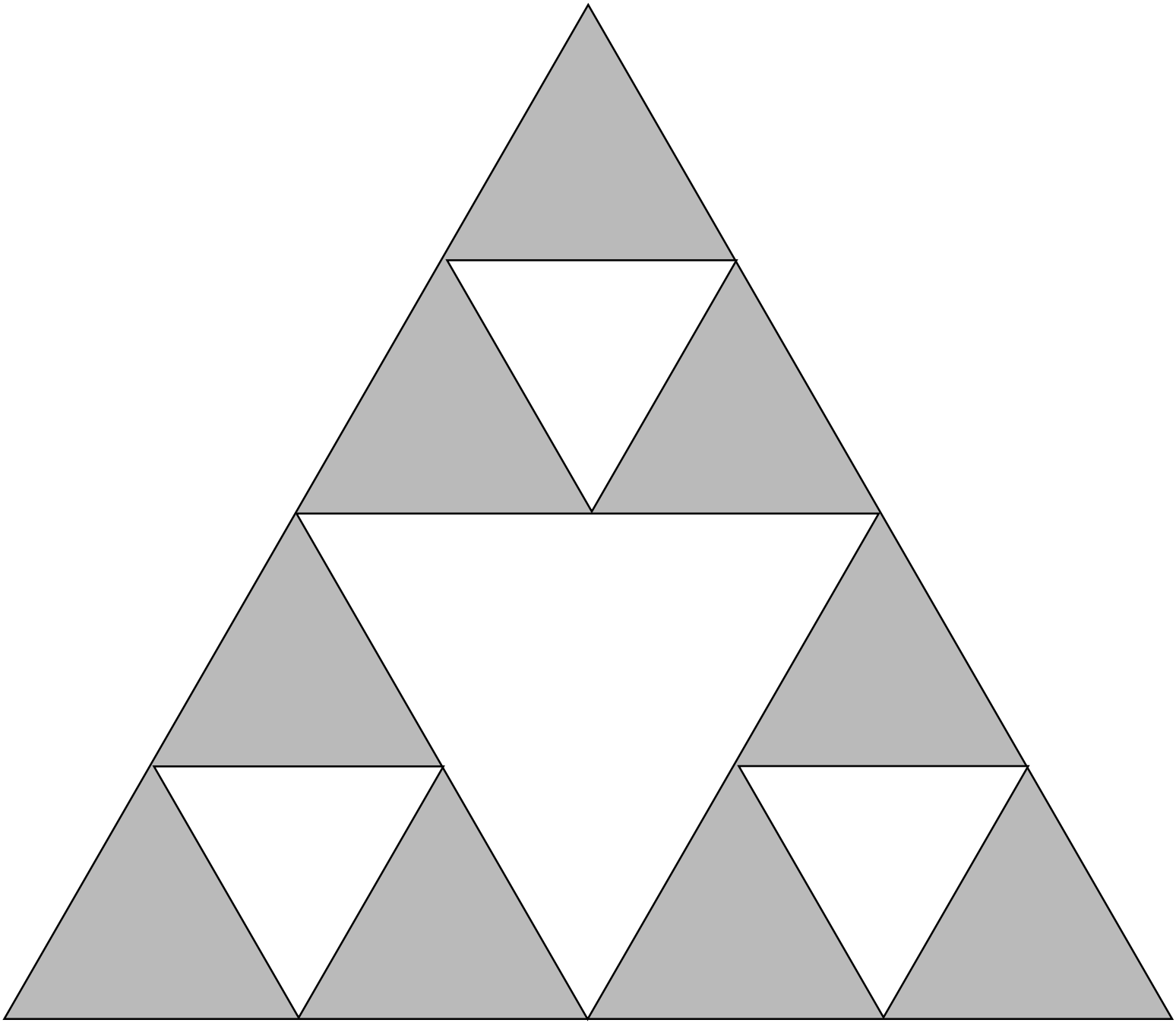
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Perimeter of the Sierpinski Triangle (Stage 1)



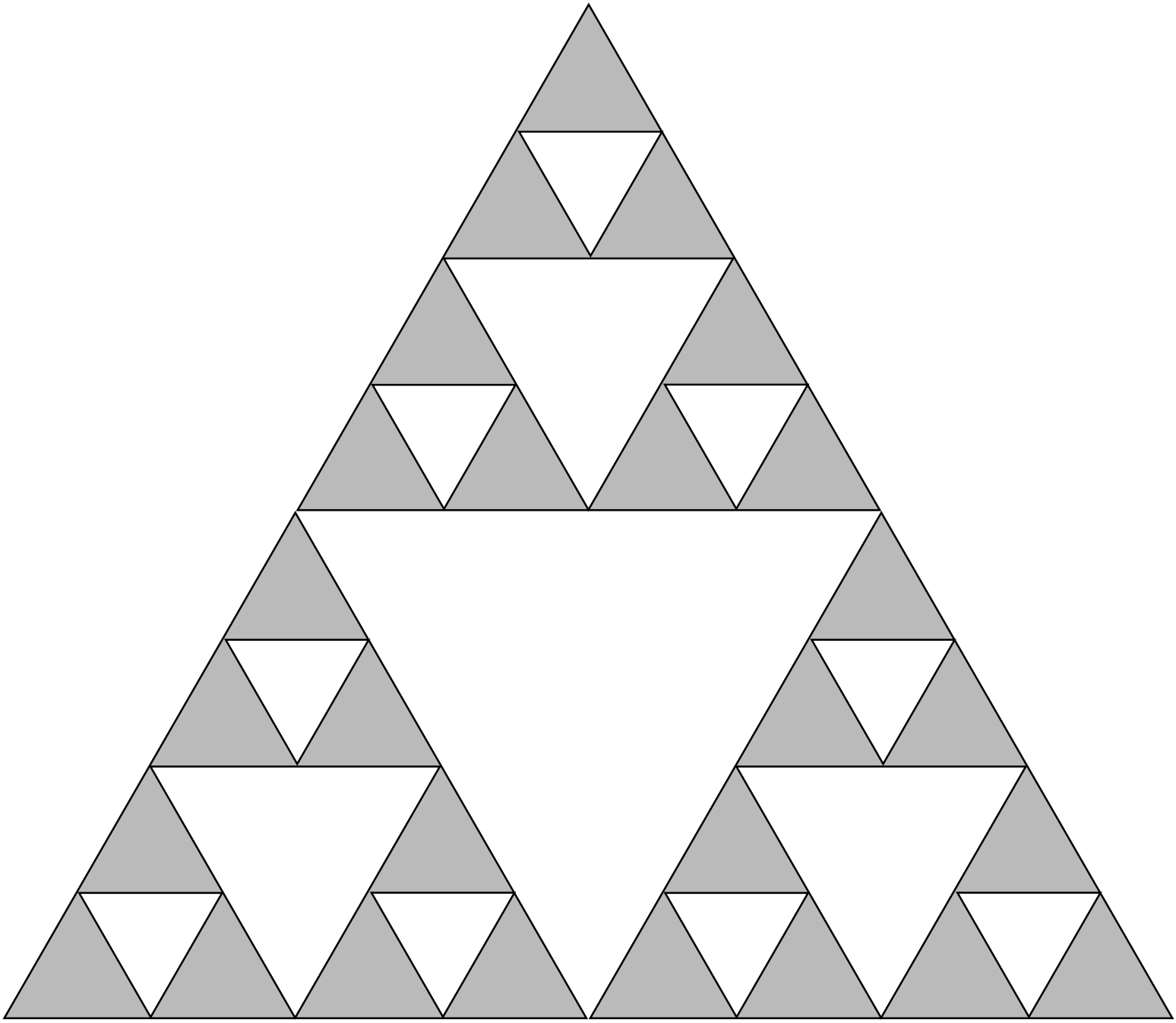
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Perimeter of the Sierpinski Triangle (Stage 2)



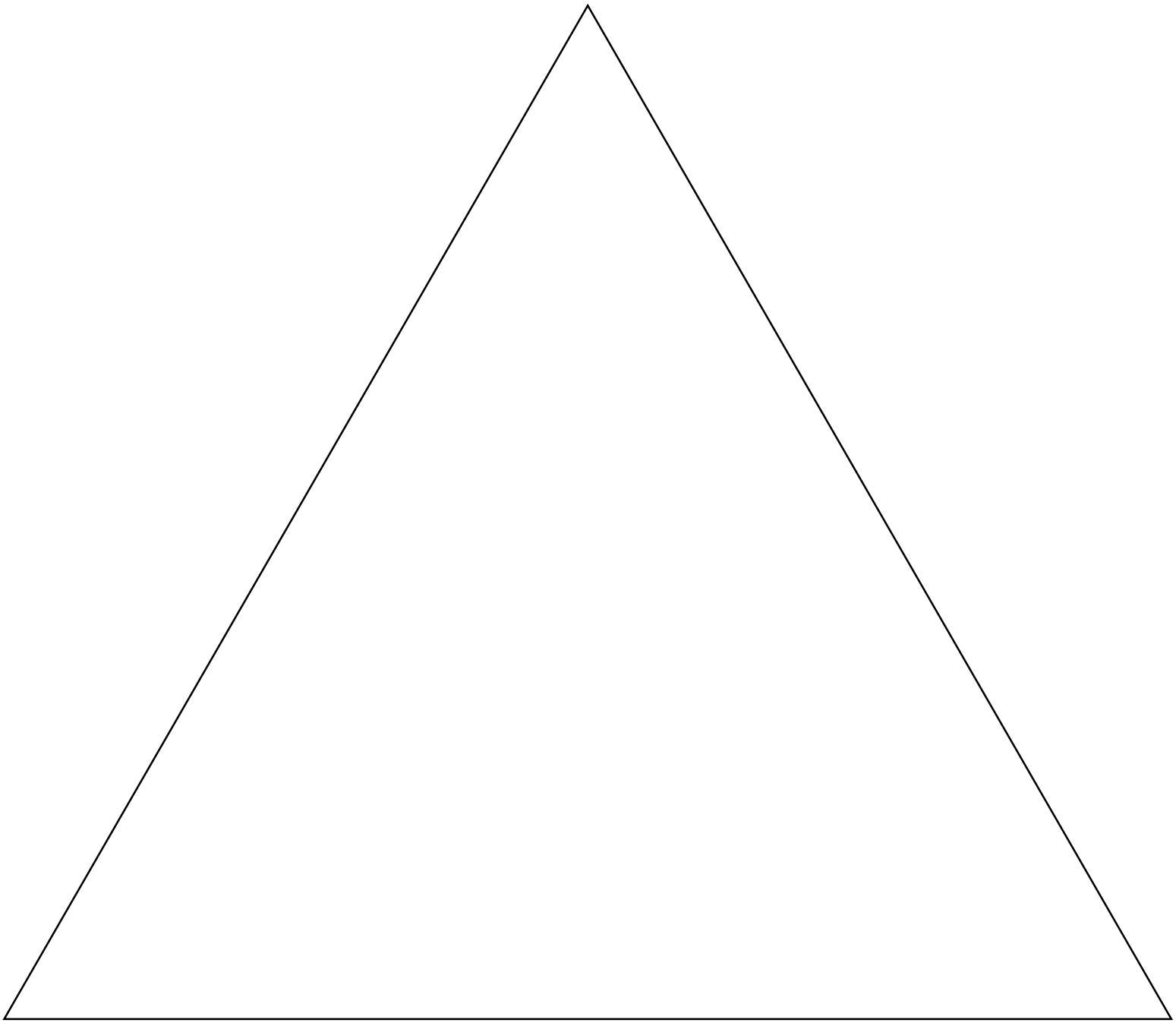
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Perimeter of the Sierpinski Triangle (Stage 3)



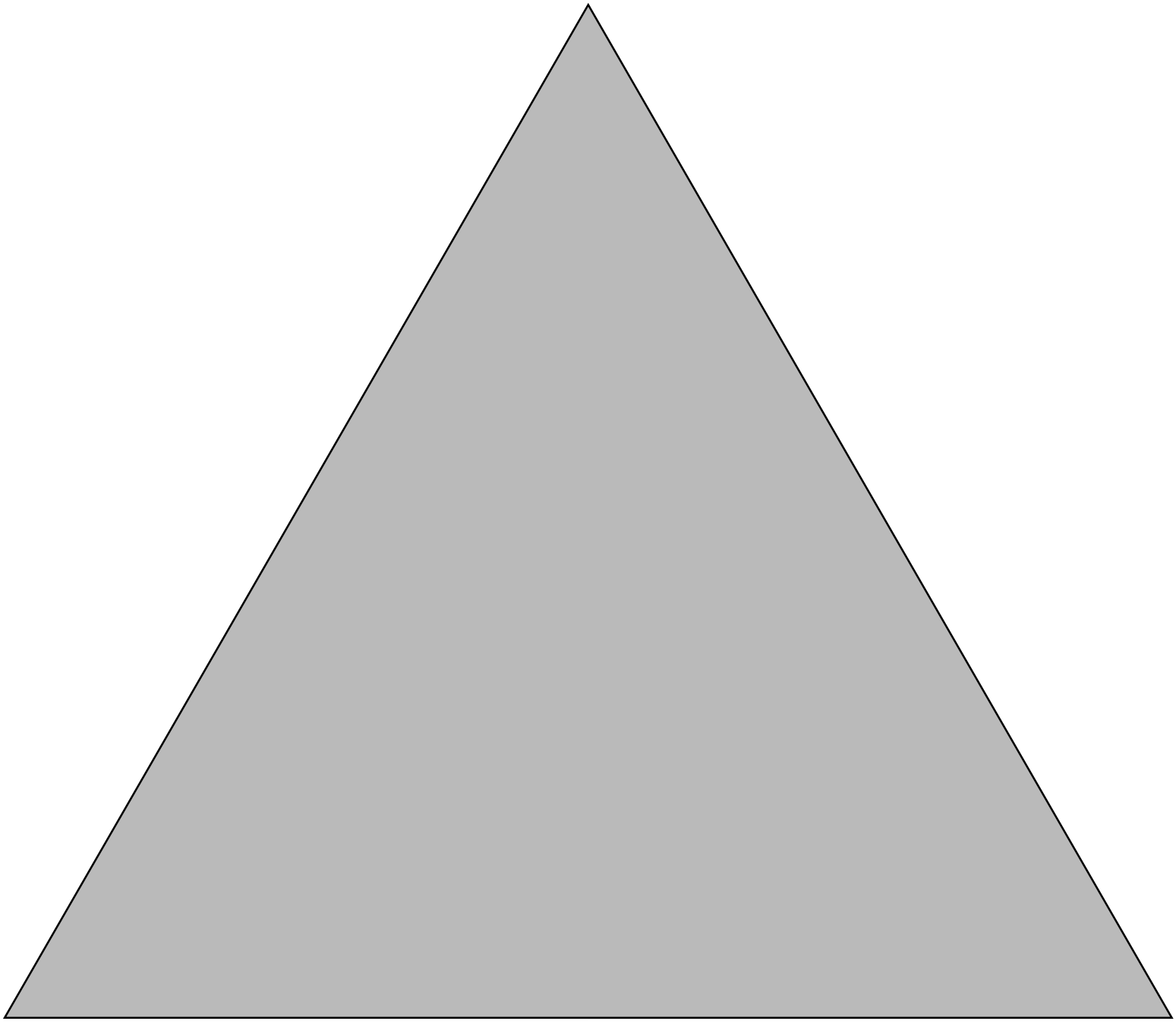
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

8-inch White Triangle Template



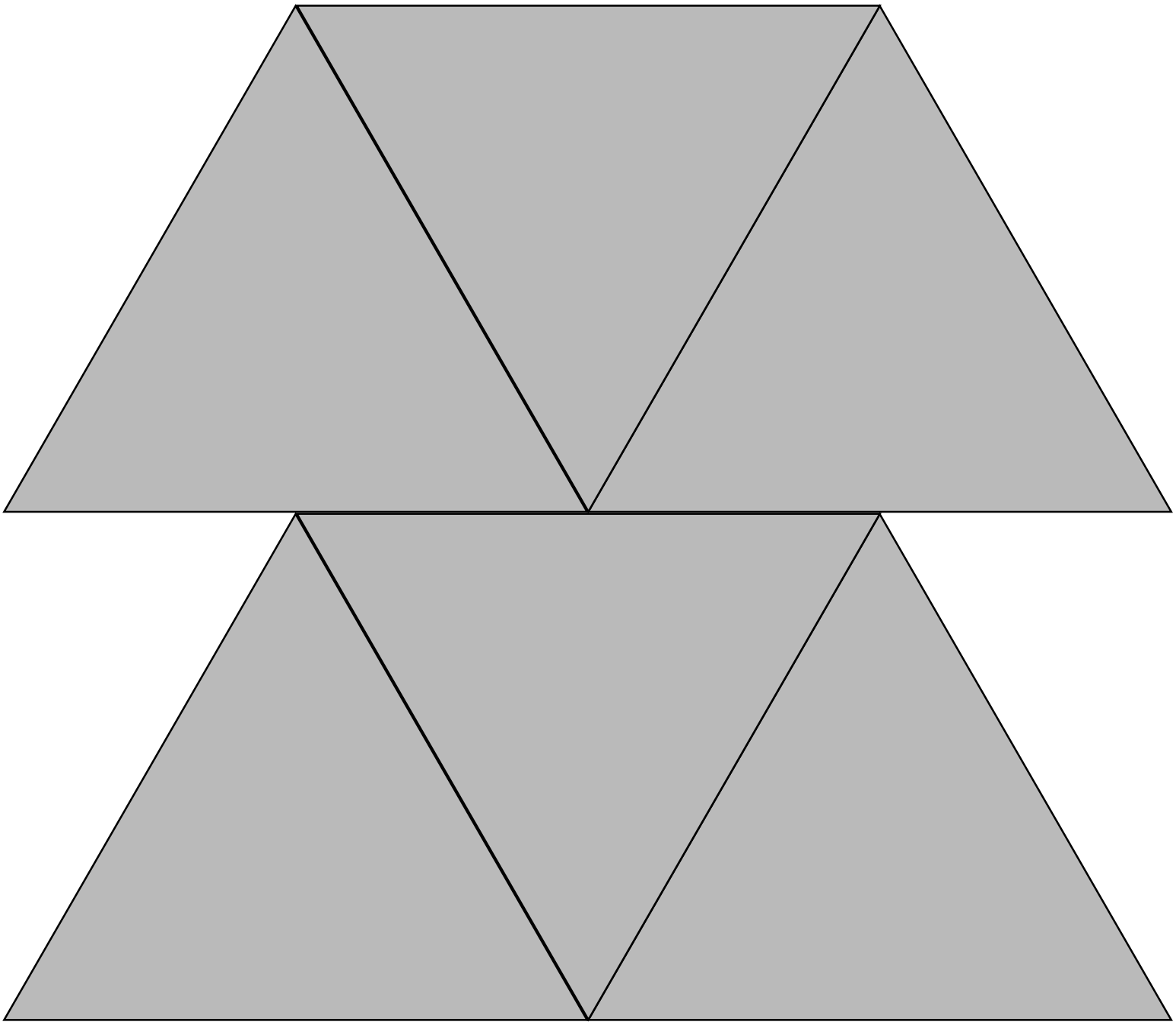
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

8-inch Shaded Triangle Template



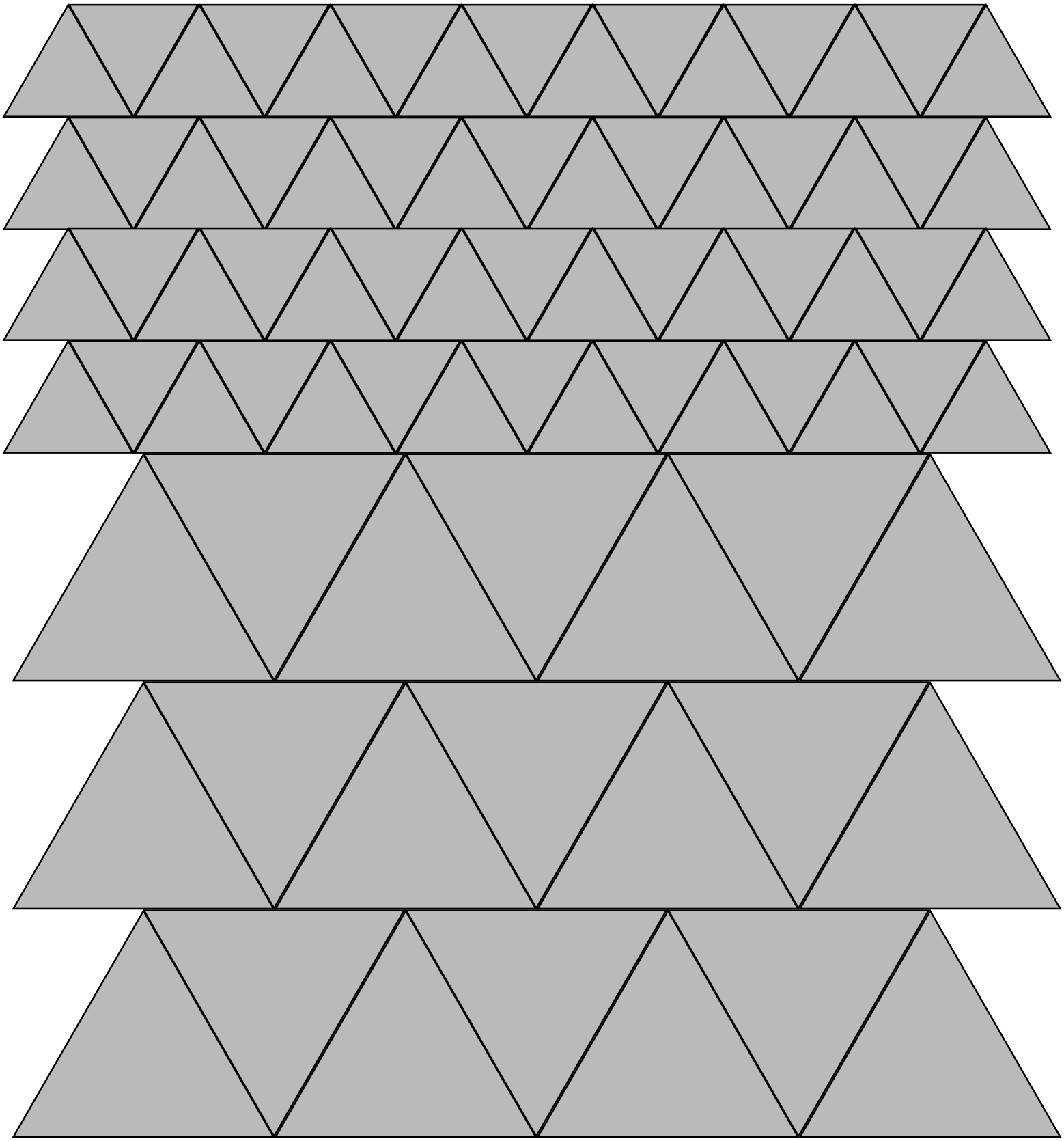
Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

4-inch Triangle Template



Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

2-inch and 1-inch Triangle Template



Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*

Developed through the National Science Foundation Project *Pattern Exploration: Integrating Mathematics and Science for the Middle Grades, 1999*